Journal of Computational Physics 228 (2009) 933–937

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Short Note

# Blending technique for compressible inflow turbulence: Algorithm localization and accuracy assessment

Johan Larsson *

*Center for Turbulence Research, Stanford University, 488 Escondido Mall, Stanford, CA 94305, United States*

ARTICLE INFO

*Article history:*
Received 12 March 2008
Received in revised form 13 October 2008
Accepted 18 October 2008
Available online 5 November 2008

*Keywords:*
Compressible flow
Turbulence
Inflow
LES
DNS

## 1. Introduction

The accuracy of large eddy and direct numerical simulations (LES and DNS) of spatially developing flows is dependent on the physical realism of the inflow turbulence. There are many possible ways to generate the inflow turbulence, with varying degrees of physical realism and applicability.

The methods by Batten et al. [1] and Di Mare et al. [2] are two examples of rather generally applicable methods to generate "synthetic turbulence", where the mean velocity profile, the Reynolds stress tensor, and the energy spectrum can be arbitrarily prescribed. Keating et al. [3] provided a comprehensive review of different methods, and tested some on spatially developing channel flow. They found that the lack of phase information in the synthetic turbulence at the inflow caused a development region before the resolved turbulence was accurate. The recycling technique by Lund et al. [4] is commonly used for spatially developing boundary layers. Instantaneous turbulence from within the domain is rescaled and recycled at the inflow, resulting in more realistic phase information. The method has been extended to compressible flows, with a comparative assessment given by Xu and Martin [5].

Several important problems in fluid mechanics have inflows with uniform mean velocity and isotropic turbulence; some examples include shock/turbulence interaction and by-pass transition with leading-edge effects taken into account [6]. To avoid the development region resulting from synthetic turbulence methods, one can instead pre-compute a database of isotropic turbulence to the desired state that is then convected into the domain using Taylor's hypothesis. There are two potential problems with this approach. First, Lee et al. [7] showed that Taylor's hypothesis is only valid for the hydrodynamic parts of the flow field, like vorticity and kinetic energy, but not for the acoustic part. Thus care is needed for problems where the acoustics are of primary importance. However, one should note that other inflow techniques likely

* Tel.: +1 650 723 9286.
*E-mail address:* jola@stanford.edu

0021-9991/$ - see front matter 
doi:10.1016/j.jcp.2008.10.027

suffer from this problem as well. The second potential problem, and the more prohibiting one, is that the cost of pre-computing the inflow database becomes exceedingly high if a long record in time in needed. For example, statistical convergence in by-pass transition may require an order of magnitude longer record of inflow turbulence than the domain flow-through time of the calculation [6].

One particulary appealing solution to this problem was proposed by Xiong et al. [6], who suggested that several independent realizations (or snap-shots) of isotropic turbulence be blended together along the streamwise direction to create an arbitrarily long database. Computing several smaller cases instead of one large requires much less memory, which is often the limiting factor in modern high-performance computing. In addition, the isotropy of the snap-shots can be utilized (by rotating them) to create a very long database using only a limited number of snap-shots [6]. Xiong et al. showed how to blend the velocity components such that the second-order pointwise moments are preserved, and estimated the error in the two-point correlations. Finally, they showed how the blending introduces an error in the dilatation field, and suggested that this be removed through solution of a Poisson equation. They then showed a qualitative 'proof-of-concept' without detailed assessment of the accuracy of the method. Thus the first objective of the present work is to quantitatively assess the accuracy of the blending technique. This is done by considering spatially decaying turbulence, and comparing the results both to temporally decaying turbulence as well as cases with synthetic turbulence at the inlet.

In addition, we present a simple modification of the Xiong et al. method that makes it more amenable to large-scale computing. They originally proposed to solve the Poisson equation in the full domain of the database, implying that the memory required to blend $N_f$ realizations of size $N^3$ scales as $N_f \times N^3$. Therefore, for long inflow databases, memory limitations alone may dictate that more processors are used to create the database than are used for the actual flow calculation. In the present work we approach the problem differently: by 'localizing' the Poisson system for dilatation removal, each blending region becomes independent of every other. This allows us to either perform the $N_f$ blending operations sequentially, with memory-usage scaling as $N^3$ (with a lower constant of proportionality as well), or to perform the $N_f$ blending operations in so-called 'embarrasingly parallel' fashion. It also enables additional realizations to be added to an existing database as they are required, including while the main simulation is running, which may be useful when the time required for statistical convergence is not known *a priori*.

## 2. Blending procedure

Consider the concatenation of two independent but statistically identical periodic boxes of turbulence of size $[0, 2\pi)^3$ into a larger box $[-2\pi, 2\pi) \times [0, 2\pi)^2$. Following [6], the original velocity fields $u_i^{(1)}$ and $u_i^{(2)}$ (with zero mean) are blended as

$$u_i = u_i^{(1)} \cos\alpha + u_i^{(2)} \sin\alpha - \partial_i \varphi, \quad |x_1| < l_b \tag{1}$$

where $\alpha$ is varied smoothly over the blending region of size $l_b$ and $\partial_i \varphi$ will be used later to remove the erroneous dilatation. One choice for $\alpha$ is

$$\alpha = \frac{\pi\beta}{2}, \quad \beta = \frac{1}{2} + \frac{1}{2} \sin\left(\frac{\pi x_1}{2 l_b}\right), \quad |x_1| < l_b$$

for which we note that $d\alpha/dx_1 \leqslant \pi^2/(8 l_b)$. Using (1) with neglected $\partial_i \varphi$, Xiong et al. [6] showed that the two-point correlation

$$R_{ij}(\mathbf{r}) = \langle u_i(\mathbf{x}) u_j(\mathbf{x} + \mathbf{r}) \rangle \approx \left[1 - \frac{1}{2}\left(r_1 \frac{d\alpha}{dx_1}\right)^2\right] R_{ij}^{(1)}(\mathbf{r})$$

where $R_{ij}^{(1)} = R_{ij}^{(2)}$ is the correlation function of the original fields. This result makes use of a Taylor expansion of $\cos\alpha$ and the fact that the fields are independent realizations. It shows that the blended field retains the second-order single-point statistics of the original fields as well as the transverse two-point correlations, while the streamwise two-point correlation is lowered by an amount controlled by the size of the blending region $l_b$ (through the bound on $d\alpha/dx_1$).

In addition, the gradients of the blended field are altered in a similar way, as

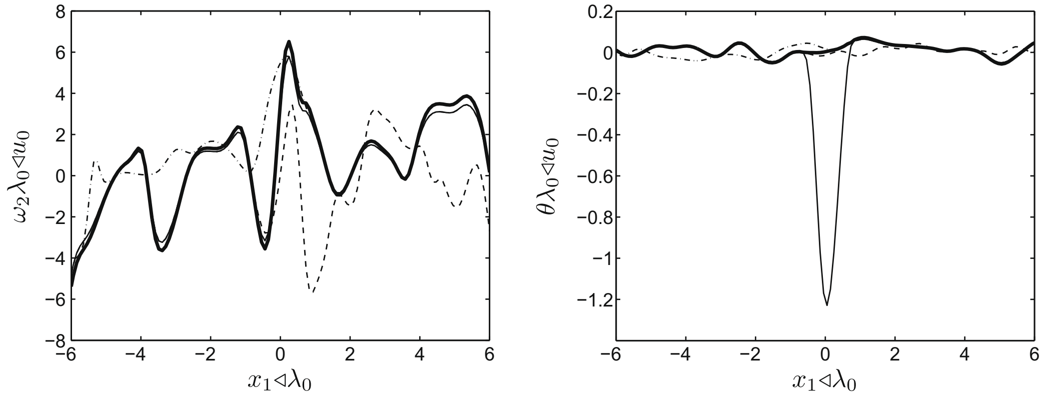
$$\partial_j u_i = \cos(\alpha) \partial_j u_i^{(1)} + \sin(\alpha) \partial_j u_i^{(2)} + \left(-u_i^{(1)} \sin\alpha + u_i^{(2)} \cos\alpha\right) \frac{d\alpha}{dx_1} \delta_{1j} - \partial_{ij}^2 \varphi \tag{2}$$

where the third term is the error due to the blending. To get a sense for this error, consider the profiles in Fig. 1.

While the blended vorticity reasonably represents the original fields, the blended dilatation has a huge peak inside the blending region. Xiong et al. [6] suggested removing the erroneous dilatation by obtaining $\varphi$ from the Poisson equation

$$\partial_{jj}^2 \varphi = q = (-u_1^{(1)} \sin\alpha + u_1^{(2)} \cos\alpha) \frac{d\alpha}{dx_1} \tag{3}$$

with Neumann and periodic boundary conditions in the streamwise and transverse directions, respectively. They therefore solved Eq. (3) in the domain of the full database $(N_f 2\pi) \times (2\pi)^2$, leading to memory-usage that increases with the length of the database. This method effectively reduces the dilatation error to within the variation of the original fields, as seen in Fig. 1.




(a) Transverse vorticity $\omega_2$. (b) Dilatation $\theta$.

**Fig. 1.** *A priori* test of blending. Profiles of the original fields (dashed and dash-dotted), blended without (thin solid) and with (thick solid) removal of dilatation. Blending size $l_b/\lambda_0 = 0.8$. Quantities normalized by rms velocity $u_0$ and Taylor length scale $\lambda_0$.

A subtle point is that the Poisson Eq. (3) only has a solution if the integral of $q$ is zero, which it is not for finite sample sizes. In practice, however, this issue is of limited importance. If the over-determined system is solved in the least-squares sense on a periodic domain, then the least-squares solution is exactly that which one would get by artificially setting the integral of $q$ to zero – which many implementations of Poisson solvers would do (implicitly) anyway since the system is singular.

### 2.1. Localization of the dilatation removal procedure

We now seek to decouple each blending region from the rest of the database as a means to decrease the memory requirements and increase the parallelism. This is accomplished by imposing the constraint

$$\partial_i \varphi = 0, \quad |x_1| \geqslant L_b \tag{4}$$

on the solutions to (3), which can be seen as solving (3) in a small region of size $2L_b$ with over-specified boundary conditions. Note that this is *not* equivalent to simply imposing Neumann conditions $\partial_1\varphi = 0$ at $|x_1| = L_b$, since this would lead to discontinuities in the transverse derivatives of $\varphi$ at that point. The system (3) and (4) is most easily solved after Fourier transforms in the transverse directions, leading to

$$(\partial_{11}^2 - k_2^2 - k_3^2)\widehat{\varphi} = \widehat{q}, \tag{5a}$$

$$\partial_1 \widehat{\varphi} = 0, \quad |x_1| = L_b, \tag{5b}$$

$$\widehat{\varphi} = 0, \quad |x_1| = L_b, \quad (k_2, k_3) \neq (0,0). \tag{5c}$$

For fixed transverse wavenumbers $(k_2,k_3)\neq(0,0)$ this can be written as $A\phi = r$, where $\phi$ and $r$ are vectors of $\widehat{\varphi}$ and $\widehat{q}$ along the streamwise direction and $A$ is the matrix approximating $(\partial_{11}^2 - k_2^2 - k_3^2)$. Next we use (5b) and (5c) to eliminate rows in the system, leading to $A$ being non-square (of size $n \times (n-2)$, say). Note that this leads to the constraints being satisfied exactly, preventing discontinuities in $\partial_j\varphi$. The (unique) least-squares solution to this over-determined system is given by

$$A^T A\phi = A^T r \tag{6}$$

Analogously, the $(k_2,k_3) = (0,0)$ mode leads to a system of form $B\phi = r$, with a least-squares solution (the compatibility condition $\int r dx_1 = 0$ is not necessarily satisfied) given by

$$(B^T B + ee^T)\phi = B^T r, \quad e^T = (1,\dots,1) \tag{7}$$

Note that the $ee^T$ part is one of many ways to ensure that the solution has zero mean – this works well with Gaussian elimination, but is not efficient for sparse-matrix solvers.

Thus the constrained minimization problem can be solved uniquely using (6) and (7). We emphasize that each blending region, i.e. each linking together of two snap-shots of turbulence, is independent of the rest of the database, which is the key to the improved parallelism and lower memory requirements.

### 2.2. Blending of fields with non-zero mean

The blending (1) does not preserve means, so when blending fields with non-zero means this should be modified to

$$f = (1-\beta)\langle f_1\rangle + \beta\langle f_2\rangle + f_1' \cos\alpha + f_2' \sin\alpha, \quad |x_1| < l_b \tag{8}$$

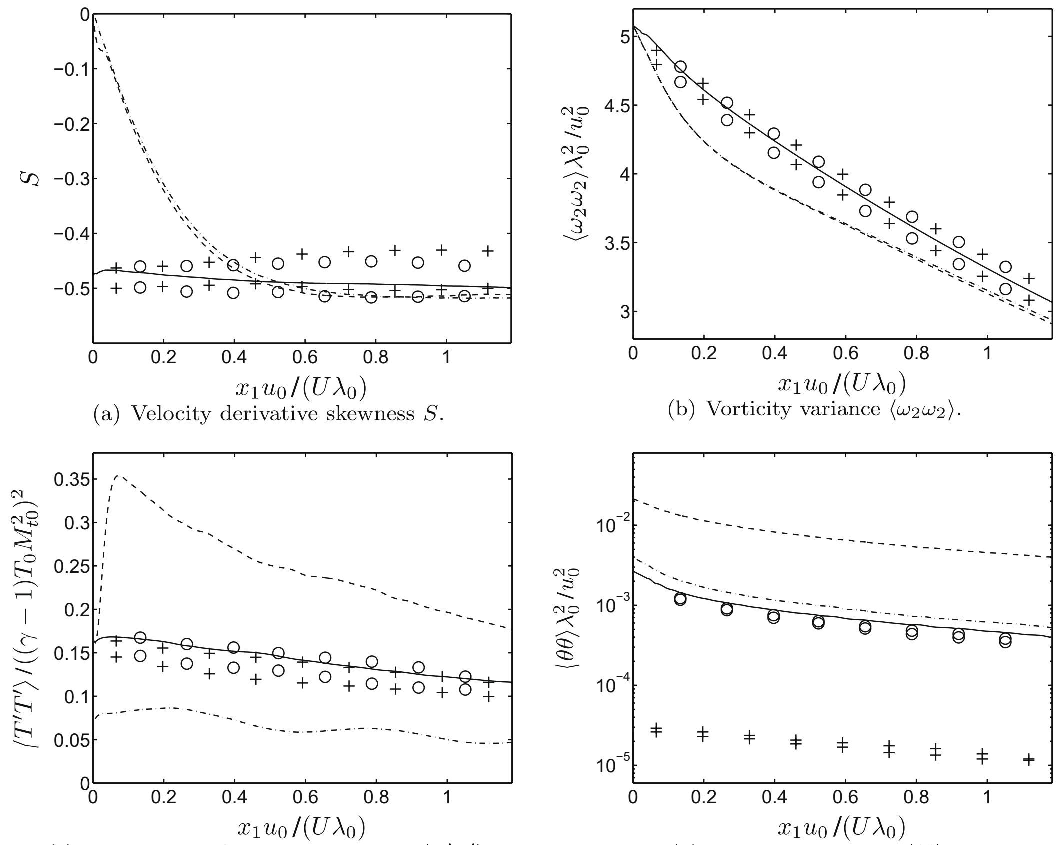
where $f = \langle f \rangle + f'$ is any quantity decomposed in its mean (e.g., spanwise average) and fluctuating component, and the subscripts denote the independent fields. This blending preserves both first- and second-order statistics, and is used to blend density and pressure in the present method.

## 3. Results

The inflow methodology is tested by computing spatially decaying turbulence in a domain $[0,4\pi] \times [0,2\pi)^2$ with $256 \times 128^2$ grid points. The mean flow Mach number is $M = U/c = 2.0$, where $U$ is the mean streamwise velocity and $c$ is the speed of sound. At the inlet the turbulence Mach number is $M_t = \sqrt{\langle u_i u_i \rangle}/c \approx 0.086$ and the Reynolds number is $Re_\lambda = \rho\lambda\sqrt{\langle u_i u_i \rangle/3}/\mu \approx 30$, where $\lambda = \sqrt{\langle u_2 u_2 \rangle / \langle \partial_2 u_2 \partial_2 u_2 \rangle}$ is the Taylor length scale. The grid resolution is sufficient to ensure resolution of the viscous dissipation. For reference we use results from several realizations of temporally decaying turbulence, transformed into a convecting frame. The initial 3 eddy turn-over times are discarded to ensure well developed turbulence; the state at this time is the target state for the spatially decaying cases. Quantities at this target state are also used (with subscript 0) to non-dimensionalize the results.

The present method is used to blend four well developed snap-shots of turbulence near the reference state into an inflow database, with blending parameters $(l_b/\lambda_0, L_b/\lambda_0) = (1, 12)$. These parameters are given in relation to the Taylor length scale, since velocity gradients scale as $u/\lambda_0$ while their error scales as $u/l_b$. The results are not very sensitive to the values of $l_b$ and $L_b$, provided that the latter is chosen large enough.

Many generally applicable synthetic turbulence methods (e.g. [1,2,7]) become essentially identical when applied to isotropic turbulence in a uniform mean flow, in that they all amount to generating random velocity fields with prescribed energy spectra. For comparison, we consider two variants here. In both cases the velocity field is randomized using the reference spectrum (which ensures the correct $Re_\lambda, M_t$, length scale, etc). In the first variant the density and temperature fields are randomized such that they agree with the reference spectra, while in the second variant these fields are found




(a) Velocity derivative skewness $S$.

(b) Vorticity variance $\langle \omega_2\omega_2 \rangle$.

(c) Temperature fluctuation variance $\langle T'T' \rangle$.

(d) Dilatation variance $\langle \theta\theta \rangle$.

**Fig. 2.** Spatially decaying turbulence with inflow database blended by $l_b/\lambda_0 = 1$ and $L_b/\lambda_0 = 12$ (solid). This is compared to a synthetic turbulence method [1,2,7] with matched spectrum of velocity, where the density and temperature fields are either: randomized to match the spectra (dashed); or solved for in low-$M_t$ limit assuming isentropy [8] (dash-dotted). For reference the extrema (min/max) of several runs are also shown: using single realizations at the inflow, i.e. without blending errors (circles); temporally decaying turbulence in convecting frame, i.e. without errors related to Taylor's hypothesis (plusses). Quantities are normalized by values at the target inflow state (subscript 0).

using the low-$M_t$ theory by Ristorcelli and Blaisdell [8]. Note that the former ensures the correct variances of density and temperature at the inlet, while the latter leads to a more consistent pressure field. Some key results from all methods are shown in Fig. 2.

The velocity derivative skewness $S = \langle(\partial_2 u_2)^3\rangle / \langle(\partial_2 u_2)^2\rangle^{3/2}$ is known to have a value around $-0.5$ in realistic turbulence (cf. Lee et al. [7]). The synthetic methods only yield realistic turbulence after half an eddy turn-over time, whereas the present method produces realistic turbulence immediately at the inlet. This carries over to the vorticity variance (which is essentially proportional to the viscous dissipation): it is accurate throughout the domain with the present method, but underpredicted when using synthetic inflow turbulence.

The temperature fluctuations are especially sensitive to the inlet condition. While the present method yields accurate results throughout the domain, the two variants of synthetic turbulence clearly do not. The lack of physical realism in the first variant produces a temperature variance that is double the correct value, despite it having the correct temperature spectrum at the inlet. The second variant instead underpredicts $\langle T'T' \rangle$ by a factor of 2, which may be due to the assumption of isentropy in the low-$M_t$ theory [8]. These results are especially important if heat transfer or combustion are of interest.

There are two potential sources of error when using the blended databases to define the inflow condition: errors introduced by the blending, and errors introduced through the use of Taylor's hypothesis at the inlet (which all methods suffer from). To separate these errors, results from using single snap-shots as inflow databases (i.e., without any blending) are also shown in Fig. 2. The Taylor-related errors are small for most quantities, but show up in the dilatation variance, where the temporally decaying turbulence has more than an order of magnitude lower values. The present method (i.e., a blended database) yields results similar to when single snap-shots are used at the inlet, which shows that the blending technique does not add additional errors. The synthetic inflow turbulence yields a similar dilatation variance when the pressure is solved for, but much higher when it is not.

## 4. Summary

The method by Xiong et al. [6] to blend several independent realizations of compressible turbulence into a longer database has been modified by re-defining the dilatation removal process as a constrained minimization problem, which is then solved using a least-squares approach that ensures a mathematically well posed problem. This modification both makes the method inherently parallel and decreases its memory requirements, which allows for very efficient generation of arbitrarily long inflow databases. For example, the databases discussed here were blended on a laptop computer.

Secondly, the method is assessed by computing spatially decaying turbulence. This quantitative assessment is an important step, given the single qualitative 'proof-of-concept' test in Ref. [6]. The present method yields realistic and accurate turbulence immediately from the inlet, without need for a development region. The only quantity that differs from temporally decaying turbulence is the dilatation, which is due to the inapplicability of Taylor's hypothesis for acoustic motions [7]. The method is compared to two variants of a more generally applicable synthetic turbulence inflow technique, with the blending technique clearly yielding superior results. This is to be expected, given that the present method both requires more computational effort (for the pre-cursor calculations) and has a limited range of validity. Nevertheless, the results illustrate the increase in accuracy that can be gained, especially for the thermodynamic fluctuations.

Finally, while only uniform inflows of isotropic turbulence are considered here, the method can be extended to more general problems. For spatially developing mixing layers, for example, one could envision a set of snap-shots of temporal mixing layers that are blended together in a similar way. This could be done by simply including the blending of the means in (8) into (1). In addition, the boundary conditions for the Poisson equation would change in the non-periodic direction.

## Acknowledgments

Financial support has been provided by the DOE SciDAC program, as well as the Natural Sciences and Engineering Research Council of Canada. Helpful suggestions from Sanjiva K. Lele and Shashank are gratefully acknowledged.

## References

[1] P. Batten, U. Goldberg, S. Chakravarthy, Interfacing statistical turbulence closures with large-eddy simulation, AIAA J. 42 (3) (2004) 485–492.
[2] L. di Mare, M. Klein, W.P. Jones, J. Janicka, Synthetic turbulence inflow conditions for large-eddy simulation, Phys. Fluids 18 (2006) 025107.
[3] A. Keating, U. Piomelli, A dynamic stochastic forcing method as a wall-layer model for large-eddy simulation, J. Turbul. 7 (2006) N12.
[4] T.S. Lund, X. Wu, K.D. Squires, Generation of turbulent inflow data for spatially-developing boundary layer simulations, J. Comput. Phys. 140 (1998) 233–258.
[5] S. Xu, M.P. Martin, Assessment of inflow boundary conditions for compressible turbulent boundary layers, Phys. Fluids 16 (7) (2004) 2623–2639.
[6] Z. Xiong, S. Nagarajan, S.K. Lele, Simple method for generating inflow turbulence, AIAA J. 42 (10) (2004) 2164–2166.
[7] S. Lee, S.K. Lele, P. Moin, Simulation of spatially evolving turbulence and the applicability of Taylor's hypothesis in compressible flow, Phys. Fluids 4 (7) (1992) 1521–1530.
[8] J.R. Ristorcelli, G.A. Blaisdell, Consistent initial conditions for the DNS of compressible turbulence, Phys. Fluids 9 (1) (1997) 4–6.